

New MDS or near MDS self-dual codes over finite fields

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Abstract: The study of MDS self-dual codes has attracted lots of attention in recent years. There are many papers on determining existence of q -ary MDS self-dual codes for various lengths. There are not existence of q -ary MDS self-dual codes of some lengths, even these lengths $< q$. We generalize MDS Euclidean self-dual codes to near MDS Euclidean self-dual codes and near MDS isodual codes. And we obtain many new near MDS isodual codes from extended negacyclic duadic codes and we obtain many new MDS Euclidean self-dual codes from MDS Euclidean self-dual codes. We generalize MDS Hermitian self-dual codes to near MDS Hermitian self-dual codes. We obtain near MDS Hermitian self-dual codes from extended negacyclic duadic codes and from MDS Hermitian self-dual codes.

Keywords: MDS codes, near MDS codes, almost MDS codes, self-dual codes, isodual codes, extended negacyclic duadic codes.

1 Introduction

Let \mathbb{F}_q denote a finite field with q elements. An $[n, k, d]$ linear code C over \mathbb{F}_q is a k -dimensional subspace of \mathbb{F}_q^n . the Singleton bound states a relationship among n , k and d : $d \leq n - k + 1$. So the Singleton defect of a q -ary linear $[n, k, d]_q$ code C is defined by $s(C) = n - k + 1 - d$, $s(C) \geq 0$.

$s(C) = 0$, C is called an MDS code. MDS codes have very good properties and are important. For examples Reed-Solomon codes are MDS codes. But for an MDS code, $n \leq k + q$ and

Main conjecture on MDS codes^[6]: For a nontrivial $[n, k, n - k + 1]$ MDS code, we have that $n \leq q + 2$ if q is even and $k = 3$ or $k = q - 1$, and $n \leq q + 1$ otherwise.

$s(C) = 1$, C is called an almost MDS code.^[6] $s(C) = s(C^\perp) = 1$, C is called a near MDS code,^[8] where C^\perp is the dual of C , defined as

$$C^\perp := \left\{ x \in \mathbb{F}_q^n : \sum_{i=1}^n x_i y_i = 0, \forall y \in C \right\}.$$

Near MDS codes and almost MDS codes have many good properties as MDS codes. There are many papers on near MDS codes and almost MDS codes.^{[1][2][6][8][9][19]}

If C satisfies $C = C^\perp$, C is called Euclidean self-dual. If C permutationally and monomially is equivalent to C^\perp , C is called isodual. All negacyclic self-dual codes, some well-known Hermitian self-dual and MDS codes are isodual.^[5] And isodual codes are formally self-dual.^[14]

If $q = r^2$, the Hermitian dual code $C^{\perp H}$ of C is defined as

$$C^{\perp H} := \left\{ x \in \mathbb{F}_{r^2}^n : \sum_{i=1}^n x_i y_i^r = 0, \forall y \in C \right\}.$$

If $C = C^{\perp H}$, C is called Hermitian self-dual. There are many papers discussing Hermitian self-dual codes.^{[7][16][18][20]} If C is MDS and Euclidean self-dual or Hermitian self-dual, C is called an MDS Euclidean self-dual code or an MDS Hermitian self-dual code, respectively. In recent years, study of MDS self-dual codes has attracted a lot of attention.^{[1][10][11][12][13][15][16][17][18]} One of these problems in this topic is to determine existence of MDS self-dual codes. When $2|q$, Grassl and Gulliver completely solve the existence of MDS Euclidean self-dual codes in [11]. In [12], Guenda obtain some new MDS Euclidean self-dual codes and MDS Hermitian self-dual codes. In [15], Jin and Xing obtain some new MDS Euclidean self-dual codes from generalized Reed-Solomon codes. In [18], Tong obtain many new MDS Euclidean self-dual codes from extended cyclic duadic codes and new MDS Hermitian self-dual codes from generalized Reed-Solomon codes or constacyclic codes. But there are many MDS self-dual codes are not existence. For examples, a $[12, 6, 7]$ MDS self-dual code over \mathbb{F}_{13} is not existence.^[11] There is not existence of a $[4, 2, 3]$ MDS Hermitian self-dual code over \mathbb{F}_4 ,^[14] and there is no MDS Hermitian self-dual code $[8, 4, 5]$ over \mathbb{F}_{16} .^[11]

In this paper, we generalize these notations of MDS self-dual codes. If C is near MDS and isodual, we call C a near MDS isodual code. If C is a near MDS code and Hermitian self-dual, we call C a near MDS Hermitian self-dual code. And we obtain them from extended negacyclic duadic codes. We also obtain near MDS Euclidean self-dual codes, which are near MDS and Euclidean self-dual, by deleting some coordinates of MDS self-dual codes. And we obtain near MDS Hermitian self-dual codes, by deleting some coordinates of MDS Hermitian self-dual codes.

2 Preliminaries

Let $(n, q) = 1$ and q be an odd prime power. The negacyclic code C over \mathbb{F}_q of length n can be considered as an ideal, $\langle g(x) \rangle$, of $R_n = \frac{\mathbb{F}_q[x]}{x^n + 1}$. Let

$$O_{2n} = \{1 + 2i | i = 0, 1, 2, \dots, n-1\}.$$

Then δ^j s ($j \in O_{2n}$) are all solutions of $x^n + 1 = 0$ over \mathbb{F}_q , where δ is a primitive $2n$ th root of unity in some extension field F of \mathbb{F}_q . The set $T \subseteq O_{2n}$ is called the defining set of C , if

$$T = \{j, j \in O_{2n} \text{ and } g(\delta^j) = 0\}.$$

Obviously, the dimension of C is $n - |T|$, and there is a constacyclic BCH bound on the minimum distance of C , which states that if T has $d - 1$ consecutive odd integers, the minimum distance of C is at least d .^{[3][4]}

Let $a \in \mathbb{F}_q^n$. Define the discrete Fourier transform (DFT) of a to be the vector $[A_0, A_1, \dots, A_n] \in F^n$, where

$$A_i = \sum_{j=0}^{n-1} a_j \delta^{(1+2i)j}, \quad 0 \leq i \leq n-1.$$

And $A_i = a(\delta^{(1+2i)})$, where $\text{ord} \delta = 2n$. Define

$$A(z) = \sum_{i=0}^{n-1} A_i z^i.$$

Lemma 1^[4] Let

$$\theta : R_n \rightarrow F^n$$

be the negacyclic DFT map defined by $\theta(a(x)) = [A_0, A_1, \dots, A_{n-1}]$. Suppose $a(x), b(x) \in R_n$. Then

- (1) θ is a ring homomorphism.
- (2) $A_i^q = A_{(qi + \frac{q-1}{2})}$.
- (3) If $0 \leq t \leq n-1$, then

$$a_t = \frac{1}{n} \delta^{-t} \sum_{i=0}^{n-1} A_i \zeta^{-it} = \frac{1}{n} \delta^{-t} A(\zeta^{-t}),$$

where $\zeta = \delta^2$.

$$(4) \sum_{t=0}^{n-1} a_t b_t = \frac{1}{n} \sum_{i=0}^{n-1} A_i B_{-i-1}.$$

(All subscripts are calculated modulo n .)

Definition 1^[4] A q -splitting of n is a multiplier μ_s of n that induces a partition of O_{2n} such that

- (1) $O_{2n} = A \cup B \cup X$.
- (2) A, B and X are unions of q -clotomic cosets.
- (3) $\mu_s(A) = B, \mu_s(B) = A$ and $\mu_s(X) = X$.

A q -splitting is of Type *I* if $X = \emptyset$. A q -splitting is of Type *II* if $X = \{\frac{n}{2}, \frac{3n}{2}\}$.

3 Euclidean isodual Codes

First we consider near MDS isodual codes.

Lemma 2^[4] If p, q are distinct odd primes, $q \equiv -1 \pmod{4}$, and r is the order of q modulo $2p^t$, then

(1) μ_{-1} gives a splitting of $2p^t$ of Type *II* if and only if $r \not\equiv 2 \pmod{4}$, in which case

$$x^{2p^t} + 1 = \lambda A(x) \tilde{A}(x)(x^2 + 1)$$

for some $\lambda \in \mathbb{F}_q$, $A(x) \in \mathbb{F}[x]$, where $\tilde{A}(x) = A(x^{-1}) \pmod{x^n + 1}$.

(2) μ_{2p^t+1} gives a splitting of $2p^t$ of Type *II* if and only if r is even, in which case

$$x^{2p^t} + 1 = \lambda A(x) A(-x)(x^2 + 1)$$

for some $\lambda \in \mathbb{F}_q$, $A(x) \in \mathbb{F}[x]$.

Lemma 3^[4] Let $q \equiv 3 \pmod{4}$, $n = 2p_1^{e_1} \cdots p_t^{e_t}$, where p_i s are distinct odd primes, and let a_i be an integer that gives a splitting of $2p_i^{e_i}$. Then n has a splitting of Type *II*. Moreover, this splitting is given by μ_a , where a is the unique integer in O_{2n} such that $a \equiv a_i \pmod{2p_i^{e_i}}$.

Theorem 1 Let $q \equiv 3 \pmod{4}$ and $n = 2p_1^{e_1} \cdots p_t^{e_t}$, where p_i are distinct odd primes. And $r = \text{ord}_n q$.

(1) μ_{-1} gives a splitting of n of Type *II* if and only if $r \not\equiv 2 \pmod{n}$.

(2) μ_{n+1} gives a splitting of n of Type *II* if and only if r is even.

Proof (1) (\Rightarrow) By Lemma 2, μ_{-1} gives a splitting of n of Type *II*, then μ_{-1} gives a splitting of $2p_i^{e_i}$ ($1 \leq i \leq t$) of type *II*. So $r_i (= \text{ord}_{2p_i^{e_i}} q) \not\equiv 2 \pmod{4}$. $r_i = \text{lcm} [\text{ord}_2 q = 1, \text{ord}_{p_i^{e_i}} q] = \text{ord}_{p_i^{e_i}} q$. So

$$r = \text{ord}_n q = \text{lcm}[1, r_1, r_2, \dots, r_t] \not\equiv 2 \pmod{4}.$$

(\Leftarrow) Let $r_i = \text{ord}_{2p_i^{e_i}} q$ ($1 \leq i \leq t$), $q^r \equiv 1 \pmod{n}$. Then $q^r \equiv 1 \pmod{2p_i^{e_i}}$. So $r_i | r$.

If $2 \nmid r$, $2 \nmid r_i$.

If $4 | r$, $n | q^r - 1$. $n \nmid q^{\frac{r}{2}} - 1$ and $n | q^{\frac{r}{2}} + 1$. If $r_i \equiv 2 \pmod{4}$. $r_i | r$, so $r_i | \frac{r}{2}$.

$$2p_i^{e_i} | q^{\frac{r}{2}} - 1, \text{ and } 2p_i^{e_i} | q^{\frac{r}{2}} + 1.$$

But it is impossible, because $(q^{\frac{r}{2}} - 1, q^{\frac{r}{2}} + 1) = 2$ and $p_i \geq 3$.

So

$$r_i \not\equiv 2 \pmod{4}, \quad i = 1, 2, \dots, t.$$

μ_{-1} gives the splitting of $2p_i^{e_i}$ of type *II* by Lemma 2. By Lemma 3, μ_{-1} gives the splitting of n of Type *II*.

We can prove (2) similarly by Lemma 3 and Lemma 2 (2).

Lemma 4^[18] (1) Let $q \equiv 3 \pmod{4}$ and $n = 2p_1^{e_1} \cdots p_s^{e_s} p_{s+1}^{e_{s+1}} \cdots p_t^{e_t}$, where

$$p_1 \equiv \cdots \equiv p_s \equiv 3 \pmod{4}, \quad p_{s+1} \equiv \cdots \equiv p_t \equiv 1 \pmod{4}.$$

Then the equation, $2 + \gamma^2 n = 0$, has a solution in \mathbb{F}_q if and only if $\sum_{i=1}^s e_i$ is odd.

(2) Let $q \equiv 1 \pmod{4}$ and $n = 2n'$, where n' is odd. Then the equation, $2 + \gamma^2 n = 0$, has a solution in \mathbb{F}_q .

Let $c = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}_q^n$, define

$$\tilde{c} = (c_0, c_1, \dots, c_{n-1}, c_\infty, c_*) \in \mathbb{F}_q^{n+2},$$

where

$$c_\infty = \gamma \sum_{i=0}^{\frac{n-1}{2}} (-1)^i a_{2i}, \quad a_* = \gamma \sum_{i=0}^{\frac{n-1}{2}} (-1)^i a_{2i+1}.$$

Let $C \subseteq \mathbb{F}_q^n$, then $\tilde{C}(\subseteq \mathbb{F}_q^{n+2})$ is defined to be the set $\{\tilde{c}, c \in C\}$.

Lemma 5^[4] Suppose q is a prime power such that $\frac{-2}{n} = \gamma^2$ for some $\gamma \in \mathbb{F}_q^*$, and suppose that D_1, D_2 are odd-like negacyclic duadic codes with multiplier μ_s of Type II.

(1) If $s = 2n - 1$, then \tilde{D}_i is self-dual for $i = 1, 2$.

(2) If $\mu_{-1}(D_i) = D_i$ for $i = 1, 2$, then $\tilde{D}_1^\perp = \tilde{D}_2$ and $\tilde{D}_2^\perp = \tilde{D}_1$.

Theorem 2 Let $q \equiv 1 \pmod{4}$ (or $q \equiv 3 \pmod{4}$) and $n = 2n'$, where n' is odd, and $2n \mid q - 1$ (or $2n \mid q + 1$). D_1 and D_2 are negacyclic codes with defining set

$$T_1 = \left\{ 1 + 2j \mid -\frac{n-2}{4} \leq j \leq \frac{n-6}{4} \right\}$$

and

$$T_2 = \left\{ 1 + 2j \mid \frac{n+2}{4} \leq j \leq \frac{3n-6}{4} \right\},$$

respectively. Then \tilde{D}_1 and \tilde{D}_2 are $[n+2, \frac{n}{2}+1, d \geq \frac{n}{2}+1]$ (near) MDS isodual codes which are extended negacyclic codes.

Proof By definitions of T_1 and T_2

$$T_1 \cap T_2 = \emptyset \text{ and } O_{2n} = T_1 \cup T_2 \cup \left\{ \frac{n}{2}, \frac{3n}{2} \right\}.$$

$$\begin{aligned} (-1)(1+2j) &\equiv 1+2(n-1-j) \pmod{2n} \\ (n+1)(1+2j) &\equiv 1+2\left(\frac{n}{2}+j\right) \pmod{2n} \end{aligned}$$

So

$$(-1)T_i = T_i, \quad (n+1)T_i = T_{i+1 \pmod{2}}, \quad i = 1, 2.$$

Case 1. When $q \equiv 1 \pmod{4}$ and $2n \mid q - 1$.

$$C_q(1+2j) = 1+2j.$$

By the constacyclic BCH bound, D_1 and D_2 are $[n, \frac{n}{2}+1, \frac{n}{2}]$ MDS odd-like negacyclic codes. Let $a = (a_0, a_1, \dots, a_{n-1}) \in D_1$ and $\text{wt}(a) = \frac{n}{2}$.

$$a(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1} = \alpha_1(x^2) + x\alpha_2(x^2).$$

Then

$$a(\delta^{\frac{n}{2}}) = \gamma^{-1}a_{\infty} + \delta^{\frac{n}{2}}\gamma^{-1}a_{*}.$$

If $a_{\infty} = a_{*} = 0$, $a(\delta^{\frac{n}{2}}) = 0$. Then $\text{wt}(a) \geq \frac{n}{2} + 1$. So $\text{wt}(\tilde{a}) \geq \frac{n}{2} + 1$. D_1 is an $[n + 2, \frac{n}{2} + 1, d \geq \frac{n}{2} + 1]$ code. Similarly, D_2 is also an $[n + 2, \frac{n}{2} + 1, d \geq \frac{n}{2} + 1]$ code.

$$\mu_{n+1}((a_0, a_1, a_2, \dots, a_{n-1}, a_{\infty}, a_{*})) = (a_0, -a_1, a_2, \dots, -a_{n-1}, a_{\infty}, -a_{*}).$$

So

$$\mu_{n+1}(\tilde{D}_i) = \tilde{D}_{i+1 \pmod{2}}.$$

\tilde{D}_1 permutationally and monomially is equivalent to \tilde{D}_2 . By Lemma 5 (2), $\tilde{D}_1^{\perp} = \tilde{D}_2$ and $\tilde{D}_2^{\perp} = \tilde{D}_1$. So \tilde{D}_1 and \tilde{D}_2 are $[n + 2, \frac{n}{2} + 1, d \geq \frac{n}{2} + 1]$ (near) MDS isodual codes which are extended negacyclic codes.

Case 2. When $q \equiv 3 \pmod{4}$ and $2n|q + 1$.

$$C_q(1 + 2j) = -1 - 2j.$$

By the constacyclic BCH bound, D_1 and D_2 are $[n, \frac{n}{2} + 1, \frac{n}{2}]$ MDS odd-like negacyclic codes. The proof can proceed as in the first case.

Next we construct (near) MDS self-dual codes from MDS self-dual codes.

Lemma 6^[11] For every odd prime power q , there exists a self-dual MDS code of length $q + 1$ over \mathbb{F}_q .

Theorem 3 Assume that q is a power of an odd prime such that $q \equiv 1 \pmod{4}$. There is a MDS Euclidean self-dual code C over \mathbb{F}_q of length $2n$. Then there is a (near) MDS Euclidean self-dual code C over \mathbb{F}_q of length $2n - 2$.

Proof Let G be a generator matrix of C , Without loss of generality, we may assume that

$$G = (I_n | A) = (e_i | \alpha_i),$$

where e_i and α_i are the rows of I_n (= the identity matrix) and A , respectively, for $1 \leq i \leq n$.

We note that

$$\text{wt}(\alpha_i) = n, \quad \alpha_i \cdot \alpha_j = 0, \quad \alpha_i \cdot \alpha_i = -1, \quad 1 \leq i \neq j \leq n.$$

Let $c \in \mathbb{F}_q$ such that $c^2 = -1$ ($q \equiv 1 \pmod{4}$). C has the following generator matrix:

$$G_1 = \left(\begin{array}{c|c} e_1 - ce_2 & \alpha_1 - c\alpha_2 \\ e_2 & \alpha_2 \\ e_3 & \alpha_3 \\ \vdots & \vdots \\ e_n & \alpha_n \end{array} \right).$$

Deleting the first two columns and the second row of G_1 produces an $(n-1) \times (2n-2)$ matrix

$$G_2 = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & \alpha_1 - c\alpha_2 \\ & & & \alpha_3 \\ & & & \vdots \\ & I_{n-2} & & \alpha_n \end{array} \right).$$

We claim that G_2 is a generator matrix of some $[2n-2, n-1, d \geq n-1]$ near MDS Euclidean self-dual code C_2 .

Obviously, the dimension of C is $n-1$. And

$$\begin{aligned} (\alpha_1 - c\alpha_2) \cdot (\alpha_1 - c\alpha_2) &= -(c^2 + 1) = 0, \\ (\alpha_1 - c\alpha_2) \cdot \alpha_{i+1} &= \alpha_1 \cdot \alpha_{i+1} - c\alpha_2 \cdot \alpha_{i+1} = 0, \quad 2 \leq i \leq n-1, \\ 1 + \alpha_{i+1} \cdot \alpha_{i+1} &= 0, \quad 2 \leq i \leq n-1, \\ 0 + \alpha_{i+1} \cdot \alpha_{j+1} &= 0, \quad 2 \leq i \neq j \leq n-1. \end{aligned}$$

$$\begin{aligned} \text{wt}(\alpha_1 - c\alpha_2) &\geq n+1-2 = n-1, \\ \text{wt} \left(k_1(\alpha_1 - c\alpha_2) + \sum_{i=2}^{n-1} k_{i+1}\alpha_{i+1} \right) &\geq \begin{cases} n+1-|T|, & k_1 = 0, \\ n+1-|T|-2 = n-1-|T|, & k_1 \neq 0, \end{cases} \end{aligned}$$

where

$$T = \{k_{i+1} | k_{i+1} \neq 0, 2 \leq i \leq n-1\}.$$

So the minimum distance d of C_2 is $\geq n-1$. C_2 is a $[2n, n-1, d \geq n-1]$ (near) MDS Euclidean self-dual code.

From Lemma 6, there is a $[14, 7, 8]$ MDS Euclidean self-dual code over \mathbb{F}_{13} . By Theorem 3, we can obtain a $[12, 6, 6]$ near MDS Euclidean self-dual code over \mathbb{F}_{13} .

4 Hermitian Self-Dual Codes

First, we consider conditions of μ_{-q} giving a q^2 -splitting of n of Type *I* or Type *II*, where $n = 2n'$, n' is odd.

Theorem 4 Let $n = 2n'$, where $n'(> 1)$ is odd.

(1) Let $q \equiv 1 \pmod{4}$. μ_{-q} gives a q^2 -splitting of n of Type *I* and Type *II*.

(2) Let $q \equiv 3 \pmod{4}$. μ_{-q} gives a q^2 -splitting of n of Type *II* if and only if $p \nmid q^s + 1$, where p is any odd prime divisor of n and s is any odd integer.

Proof Let $n = 2n'$, where n' is odd. So $\{\frac{n}{2}, \frac{3n}{2}\} \subseteq O_{2n}$, and

$$C_{q^2} \left(\frac{n}{2} \right) = \frac{n}{2}, \quad C_{q^2} \left(\frac{3n}{2} \right) = \frac{3n}{2}.$$

(1) Let $q \equiv 1 \pmod{4}$. For some j ($0 \leq j \leq n-1$) and l ($l \geq 0$),

$$(-q)(1+2j) \equiv (q^2)^l(1+2j) \pmod{2n}.$$

Then

$$2n | (q^{2ml} + q)(1+2j) \text{ and } 4 | (q^{2ml} + q).$$

But $q^{2ml} + q \equiv 1 + 1 \equiv 2 \pmod{4}$. It is a contradiction.

So for any j ($0 \leq j \leq n-1$) and l ($l \geq 0$),

$$(-q)(1+2j) \not\equiv (q^2)^l(1+2j) \pmod{2n}.$$

And $(-q)\frac{n}{2} \equiv \frac{3n}{2} \pmod{2n}$. So μ_{-q} gives a q^2 -splitting of n of Type *I* and Type *II*.

(2) Let $q \equiv 3 \pmod{4}$,

$$(-q)\frac{n}{2} \equiv \frac{n}{2} \pmod{2n} \text{ and } (-q)\frac{3n}{2} \equiv \frac{3n}{2} \pmod{2n}.$$

So μ_{-q} can not give a q^2 -splitting of n of Type *I*.

If there is an odd prime p , where $p|n$, and odd integer l such that $p|q^l + 1$,

$$\frac{n}{2p} \in O_{2n}, \quad 1+2j_0 = \frac{n}{2p}, \text{ for some } 0 \leq j_0 \leq n-1.$$

So

$$2n | (q^{l+1} + q)(1+2j_0),$$

and

$$(-q)(1+2j_0) \equiv (q^2)^{\frac{l+1}{2}}(1+j_0) \pmod{2n}.$$

So μ_{-q} can not give a q^2 -splitting of n of Type *II*.

If $p \nmid q^s + 1$, where p is any odd prime divisor of n and s is any odd integer.

$$\begin{aligned} 2n | ((q^2)^{\frac{s+1}{2}} + 1)(1+2j) &\Leftrightarrow \frac{n}{2} | 1+2j \\ &\Leftrightarrow 1+2j = \frac{n}{2} \text{ or } \frac{3n}{2}. \end{aligned}$$

So μ_{-q} gives a q^2 -splitting of n of Type *II*.

Similarly, we can prove the next theorem.

Theorem 5 Let $n = 2n'$, where n' is odd. μ_{-1} and μ_{n+1} give q^2 -splittings of n of Type *I* and Type *II*.

Let $c = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{F}_{q^2}^n$, define

$$\bar{c} = (c_0, c_1, \dots, c_{n-1}, c_\infty, c_*) \in \mathbb{F}_{q^2}^{n+2},$$

where

$$c_\infty = \gamma \sum_{i=0}^{\frac{n-1}{2}} (-1)^i c_{2i}, \quad c_* = \gamma \sum_{i=0}^{\frac{n-1}{2}} (-1)^i c_{2i+1},$$

and γ is a solution of equation $2 + \gamma^{q+1}n = 0$ in \mathbb{F}_{q^2} . Note that the equation, $2 + \gamma^{q+1}n = 0$, always has a solution in \mathbb{F}_{q^2} .

Let $C \subseteq \mathbb{F}_{q^2}^n$, then $\overline{C}(\subseteq \mathbb{F}_{q^2}^{n+2})$ is defined to be the set $\{\tilde{c}, c \in C\}$.

Theorem 6 Let $n = 2n'$, where n' is odd. Suppose that D_1, D_2 are odd-like negacyclic duadic codes of length n over \mathbb{F}_{q^2} with multiplier μ_{-q} of Type II.

(1) \overline{D}_i is Hermitian self-dual for $i = 1, 2$.

(2) If $\mu_{-q}(D_i) = D_i$ for $i = 1, 2$, then $\overline{D}_1^{\perp H} = \overline{D}_2$ and $\overline{D}_2^{\perp H} = \overline{D}_1$.

Proof (1) Let $\bar{a}, \bar{b} \in D_i$. Let $\omega = \delta^{\frac{n}{2}}$, a primitive 4th root of unity. Define

$$\begin{aligned} a(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = \alpha_1(x^2) + x\alpha_2(x^2), \\ b(x) &= b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} = \beta_1(x^2) + x\beta_2(x^2). \end{aligned}$$

So

$$a_\infty = \gamma\alpha_1(-1), \quad a_* = \gamma\alpha_2(-1), \quad b_\infty = \gamma\beta_1(-1), \quad b_* = \gamma\beta_2(-1).$$

$$\begin{aligned} \sum_{t=0}^{n-1} a_t b_t^q &= \sum_{t=0}^{n-1} \left(\frac{1}{n} \sum_{i=0}^{n-1} A_i \delta^{-(1+2i)t} \right) \left(\frac{1}{n^q} \sum_{j=0}^{n-1} B_j^q \delta^{-(1+2j)qt} \right) \\ &= \sum_{t=0}^{n-1} \left(\frac{1}{n^{q+1}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} A_i B_j^q \delta^{-t[(1+2i)+(1+2j)q]} \right) \\ &= \frac{1}{n^q} \sum_{j=0}^{n-1} A_{(-\frac{q+1}{2}-qj)} B_j^q \quad \left(1 + 2 \left(-\frac{q+1}{2} - qj \right) = (-q)(1+2j) \right) \\ &= \begin{cases} \frac{1}{n^q} \left[B_{\frac{n-2}{4}}^q A_{\frac{3n-2}{4}} + A_{\frac{n-2}{4}} B_{\frac{3n-2}{4}}^q \right] & q \equiv 1 \pmod{4} \\ \frac{1}{n^q} \left[B_{\frac{n-2}{4}}^q A_{\frac{n-2}{4}} + A_{\frac{3n-2}{4}} B_{\frac{3n-2}{4}}^q \right] & q \equiv 3 \pmod{4} \end{cases} \\ &= \begin{cases} \frac{1}{n^q} [b^q(\omega)a(-\omega) + a(\omega)b^q(-\omega)] & q \equiv 1 \pmod{4} \\ \frac{1}{n^q} [b^q(\omega)a(\omega) + a(-\omega)b^q(-\omega)] & q \equiv 3 \pmod{4} \end{cases} \\ &= \frac{2}{n^q} [\alpha_1(-1)\beta_1^q(-1) + \alpha_2(-1)\beta_2^q(-1)] \\ &= \frac{2}{n^q} \gamma^{-1-q} [a_\infty b_\infty^q + a_* b_*^q]. \end{aligned}$$

So

$$\begin{aligned}
(\bar{a}, \bar{b}) &= \frac{2}{n^q} \gamma^{-1-q} [a_\infty b_\infty^q + a_* b_*^q] + [a_\infty b_\infty^q + a_* b_*^q] \\
&= \left(\frac{2}{n^q} \gamma^{-1-q} + 1 \right) [a_\infty b_\infty^q + a_* b_*^q] \\
&= \frac{1}{n^q} \gamma^{-1-q} (2 + n^q \gamma^{q+1}) [a_\infty b_\infty^q + a_* b_*^q] \\
&= 0.
\end{aligned}$$

Note that $2 + n^q \gamma^{q+1} = 2 + n \gamma^{q+1}$ over \mathbb{F}_{q^2} .

So \bar{D}_i is Hermitian self-dual for $i = 1, 2$.

(2) $\bar{D}_1^{\perp H} = \bar{D}_2$ and $\bar{D}_2^{\perp H} = \bar{D}_1$ can be proved similarly as (1).

Theorem 7 Let $n = 2n'$, where n' is odd. Let D is a negacyclic code with defining set

$$T = \left\{ 1 + 2j \mid \frac{n+2}{4} \leq j \leq \frac{3n-6}{4} \right\}.$$

(1) When $q \equiv 1 \pmod{4}$. Let $n|q+1$. Then \bar{D} is an $[n+2, \frac{n}{2}+1, d \geq \frac{n}{2}+1]$ (near) MDS Hermitian self-dual code which is the extended negacyclic code.

(2) When $q \equiv 3 \pmod{4}$. Let $n|q-1$. Then \bar{D} is an $[n+2, \frac{n}{2}+1, d \geq \frac{n}{2}+1]$ (near) MDS Hermitian self-dual code which is the extended negacyclic code.

Proof From $n|q+1$ or $n|q-1$, we have $2n|q^2-1$. So $C_{q^2}(1+2j) = 1+2j$ and D is an $[n+2, \frac{n}{2}+1, \frac{n}{2}]$ MDS negacyclic code.

(1) When $q \equiv 1 \pmod{4}$ and $n|q+1$. Then $q+1 = ln$, where l is odd.

$$\begin{aligned}
(-q)(1+2j) = -q - 2qj &= 1 - (q+1) - 2(q+1)j + 2j \\
&\equiv 1 + 2j - 2\frac{ln}{2} \\
&\equiv 1 + 2\left(\frac{n}{2} + j\right) \pmod{2n}.
\end{aligned}$$

So

$$(-q)T \cap T = \emptyset, \quad (-q)T \cup T = O_{2n} \setminus \left\{ \frac{n}{2}, \frac{3n}{2} \right\}.$$

And D is an odd-like negacyclic duadic code. By Theorem 6, \bar{D} is Hermitian self-dual. Just like the proof of Theorem 2, we can prove that $wt(\bar{D}) \geq \frac{n}{2} + 1$.

Because

$$(-1)T = T \quad \text{and} \quad O_{2n} = T \cup (n+1)T \cup \left\{ \frac{n}{2}, \frac{3n}{2} \right\},$$

\tilde{D} is existence by Lemma 4 (2) and \tilde{D} is isodual by Theorem 2. By construction methods of \tilde{D} and \bar{D} , \bar{D} permutationally and monomially is equivalent \tilde{D} . So \bar{D} is isodual. \bar{D} is a near MDS code.

So \overline{D} is an $[n + 2, \frac{n}{2} + 1, d \geq \frac{n}{2} + 1]$ near MDS Hermitian self-dual code which is the extended negacyclic code.

(2) When $q \equiv 3(\text{mod}4)$ and $n|q - 1$.

$$(-q)(1 + 2j) \equiv 1 + \left(\frac{n}{2} - 1 - j\right) \pmod{2n}.$$

So

$$(-q)T \cap T = \emptyset, \quad (-q)T \cup T = O_{2n} \setminus \left\{\frac{n}{2}, \frac{3n}{2}\right\}.$$

D is an odd-like negacyclic duadic code. So the proof can proceed as in the first case.

Because the equation, $1 + c^{q+1} = 0$, always has a solution in \mathbb{F}_{q^2} . Just like Theorem 3, we have the next theorem.

Theorem 8 Assume that q is a power of an odd prime. There is an MDS Hermitian self-dual code C over \mathbb{F}_{q^2} of length $2n$. Then there is a near MDS Hermitian self-dual code C over \mathbb{F}_{q^2} of length $2n - 2$.

Proof Because $C^{\perp H} = (C^q)^{\perp}$, where $C^q := \{c^q = (c_0^q, \dots, c_{n-1}^q), c \in C\}$. And C^q and C have same weighted distributions. A Hermitian self-dual code C is formally self-dual. Just like the proof of Theorem 3, we can prove the theorem.

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